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Models and Tools for Cyber-Physical Systems
Class sheet 2

Exercise 1: Modelling the Water Tank

Following the same procedure used in the lecture for the thermostat, draw a hybrid automaton for the water tank system depicted in Figure 1.

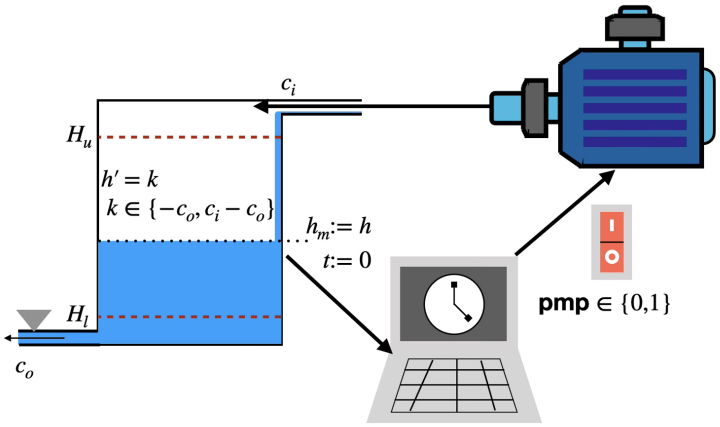

Figure 1: A water tank system to be modelled as a hybrid automaton.

Exercise 2: Modelling the Bouncing Ball

We model a ball dropped from rest as a hybrid system. The diagram in Figure 2 shows the hybrid automaton with four missing labels. Fill in the labels using the constraints below.

- The ball starts from rest, so the initial velocity is zero: $v(0) = v_0 = 0$.
- Let h_0 denote the starting height: $h(0) = h_0$.
- The rate of change of height equals the velocity, and the velocity decreases at the rate of the gravitational constant g: $\dot{h} = v, \quad \dot{v} = -g$.
- The ground is the boundary condition $h(t) \geq 0$.
- When the ball hits the ground ($h(t) = 0$), the velocity changes discretely via $v := -a \cdot v$, where $0 < a < 1$ is a damping constant.

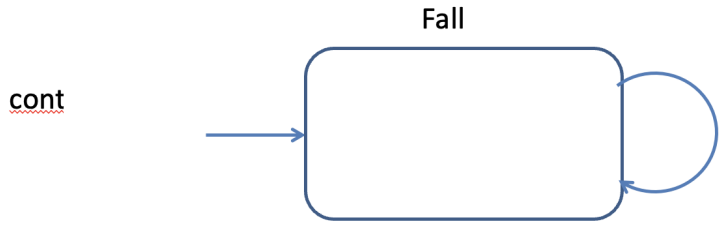

Figure 2: Hybrid automaton for the bouncing ball with four labels missing. Fill in: (1) the inputs, (2) the ODEs, (3) the invariant, (4) the transition reset.

Exercise 3: Analysing the Bouncing Ball

Although the model from Exercise 2 abstracts the collision with the ground, it hides a subtle undesired behaviour. We analyse the system's evolution over time.
The general solution to the ODE $\dot{h} = v$, $\dot{v} = -g$ is:

$$v(t) = -g \cdot t + v_{prev}, \qquad h(t) = -\tfrac{g}{2} \cdot t^2 + v_{prev} \cdot t + h_{prev}.$$

(a) How long does it take the ball to reach the floor before the first bounce? Express your answer in terms of h_0 and g.

(b) Use the result from (a) to find the velocity v_1 immediately after the first bounce, in terms of h_0, a, and g.

(c) Find the time t_{next} between two successive bounces (from one bounce to the next). *Hint:* between bounces, $h_{prev} = h_{next} = 0$. Express your answer in terms of v_{prev} and g.

(d) From (c), deduce v_{next} in terms of v_{prev}.

(e) What is the velocity v_{n+1} immediately after the $(n+1)$th bounce, in terms of the velocity v_n after the nth bounce? And in terms of the velocity v_1 after the first bounce?

(f) Using your answers from (c) and (e), find the total duration of all bounces after the first one, $\sum_{i=1}^{\infty} t_{i+1}$. *Hint:* consider t_{n+1} from (c) as expressed in terms of v_{n+1} (why?) and recall $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$.

(g) Does this make physical sense?

Exercise 4: Zeno and Non-Zeno Definitions

Complete each sentence below.

- An infinite execution $s_0 \xrightarrow{t_1} s_1 \xrightarrow{t_2} \cdots$ is called *Zeno* if the infinite sum of all durations t_i is _____.
- An infinite execution $s_0 \xrightarrow{t_1} s_1 \xrightarrow{t_2} \cdots$ is called *non-Zeno* if the infinite sum of all durations t_i _____.
- A state s of a process HP is called *Zeno* if _____ (Hint: $\forall$) execution starting in s is Zeno.
- A state s of a process HP is called *non-Zeno* if _____ (Hint: $\exists$) non-Zeno execution starting in s.
- A hybrid process HP is called *non-Zeno* if _____ (Hint: $\forall$) reachable state of HP is non-Zeno.
- In a non-Zeno system HP, it is possible for time to _____ (Hint: $t \to \infty$) at every point during one of its executions.
- In a Zeno system HP, an execution could end up in a state from which only Zeno executions are possible, that is, an execution where the total duration of all transitions is _____ (Hint: $\sum t < b$).

Exercise 5: Identifying Zeno Systems

For each system below, determine whether it is Zeno or non-Zeno. Justify your answer using the definitions from Exercise 4.

(a) The thermostat from the lecture.

(b) The bouncing ball from Exercise 2.

(c) The system in Figure 3.



Figure 3: System (c).

(d) The system in Figure 4.

(e) The system in Figure 5.

(f) *Extra:* Does the system in (d) have a Zeno execution?

(g) *Extra:* What is the total duration of all transitions in system (c)?



Figure 4: System (d).

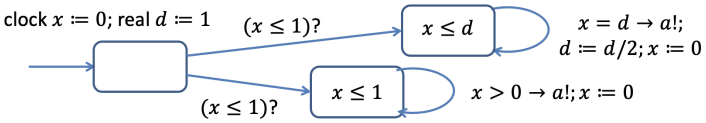

Figure 5: System (e).

Exercise 6: Why Zeno Processes Matter

Recall the following definitions: a state s is *reachable* if there exists a finite execution starting in an initial state and ending in s. A property φ is an *invariant* if all reachable states satisfy φ.

(a) Is mode B in Figure 6 reachable?

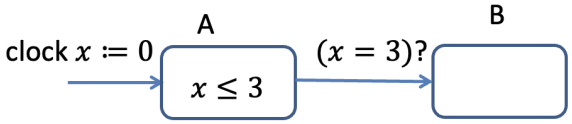

Figure 6: System for question (a).

(b) Is mode B in Figure 7 reachable?

(c) To prevent the bouncing ball from being Zeno, add a discrete command that fires when the velocity has "diminished enough": instead of bouncing, the ball comes to rest by setting $v := 0$. Extend the diagram from Exercise 2 with this modification, using e for the "small-enough" threshold.

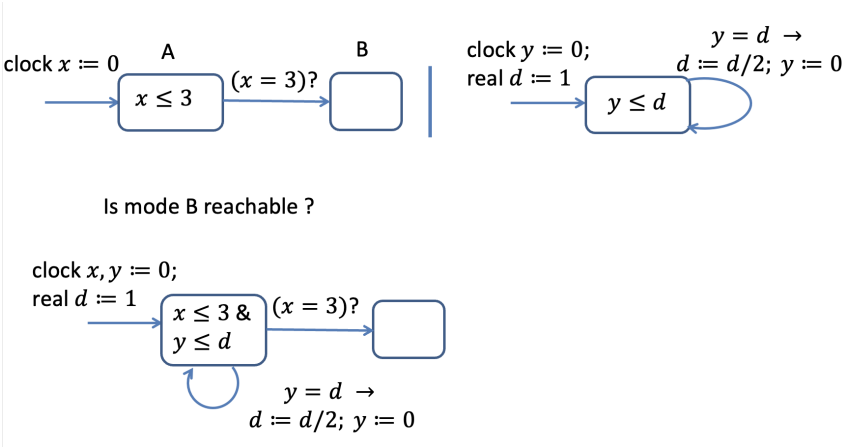

Figure 7: System for question (b): a composition involving a Zeno process.